Bootstrapping for Approximate Homomorphic Encryption

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“Word Encryption”
(BGV12, Bra12, FV12)

Packing & SIMD operations on $GF(p^d)$
between RLWE ciphertexts

Long latency (Bootstrapping)

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- Approximate Rounding is easy!

  $[\langle ct, sk\rangle]_q = M$

  HomRnd : $ct \mapsto ct' = \lceil p^{-1} \cdot ct \rfloor$

  $\Rightarrow [\langle ct', sk\rangle]_{q/p} \approx M/p$

$(1.234) \times (5.678) = (1{,}234 \times 5{,}678) \times 10^{-6} = (7{,}006{,}652) \times 10^{-6} \approx (7{,}007) \times 10^{-3}$.

Functionality of Approximate HE

Packing Technique

- $K = Q[x]/(\Phi_m(x))$, $R = Z[x]/(\Phi_m(x))$.
- $\Phi_m(X) = \prod_i(x - \zeta_i)$ for the primitive m-th roots of unity ζ_i.
- Encoding map: $(M_i)_i \mapsto M(X)$ such that $M(\zeta_i) = M_i$

Approximate addition, multiplication, and rounding

- Every homomorphic operation includes a small noise

Evaluation of Analytic Functions

- exp (z),
- z^{-1}

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Packing & SIMD operation over the real/complex numbers (add, mult + rounding) between RLWE ciphertexts

Application Researches of HE (2017~)

- Machine Learning & Neural Networks: 7
- Biomedical & Health data analysis: 3
- Bioinformatics: 3
- Genomic data analysis: 3
- Cyber Physical System & Internet of Things: 4
- Smart Grid: 3
- Image processing: 3
- Voting: 2
- Advertising: 2

> 80 %

[Kim-Song-Kim-Lee-Cheon'18] iDASH Privacy & Security Competition 2017

Six minutes to train a logistic regression model from encrypted dataset of size 1579 * (18+1).

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  How to (efficiently) evaluate the modular reduction $(q \cdot t + M) \mapsto M$?

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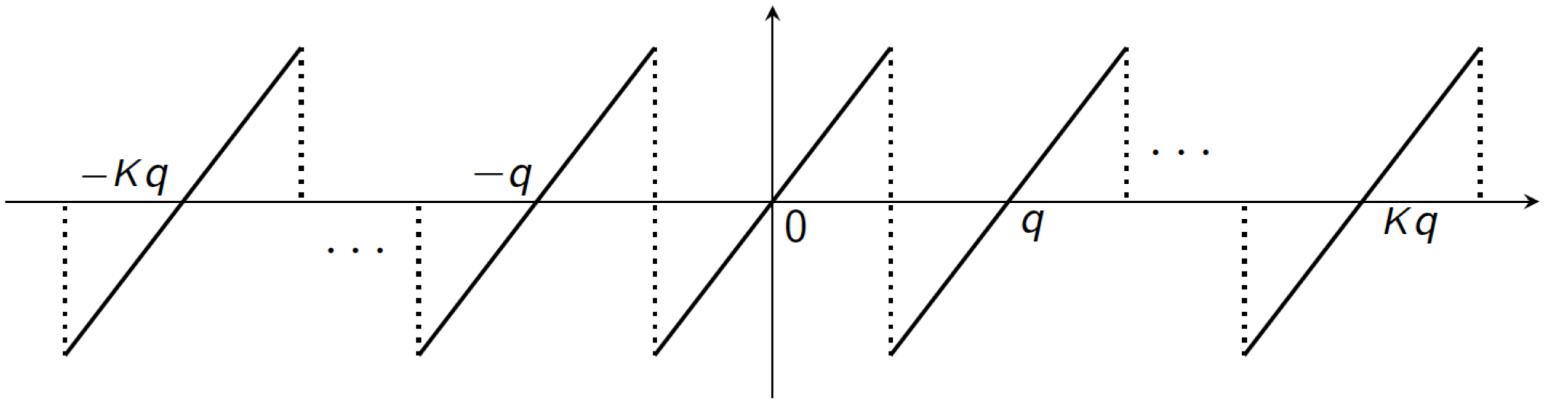
Efficiency	Degree $d = O(Kq)$, Complexity $O(d)$ operations - exp. on the depth!
Correctness	Large error on the boundary



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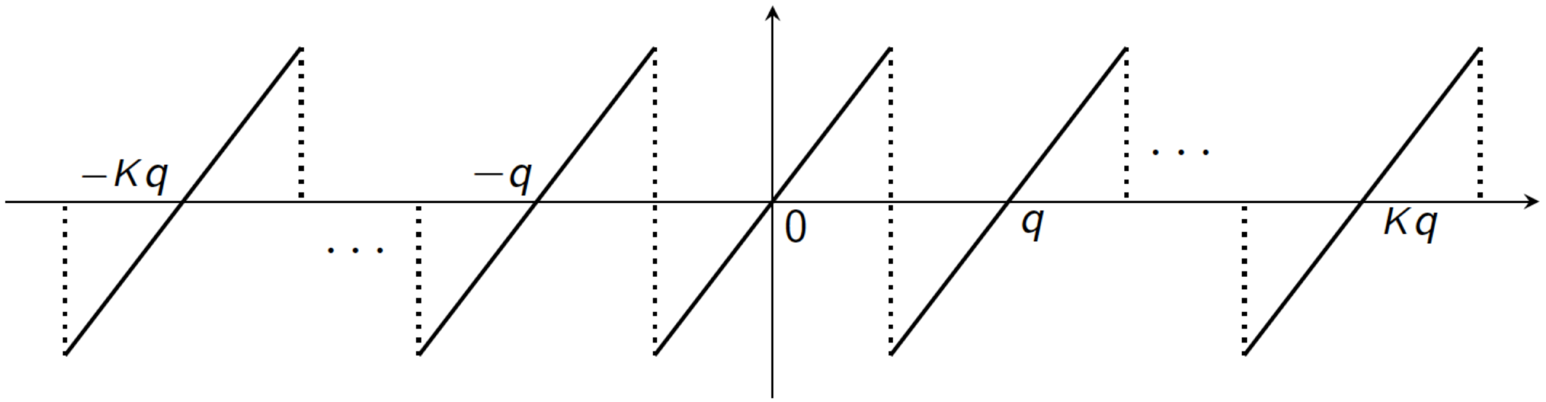

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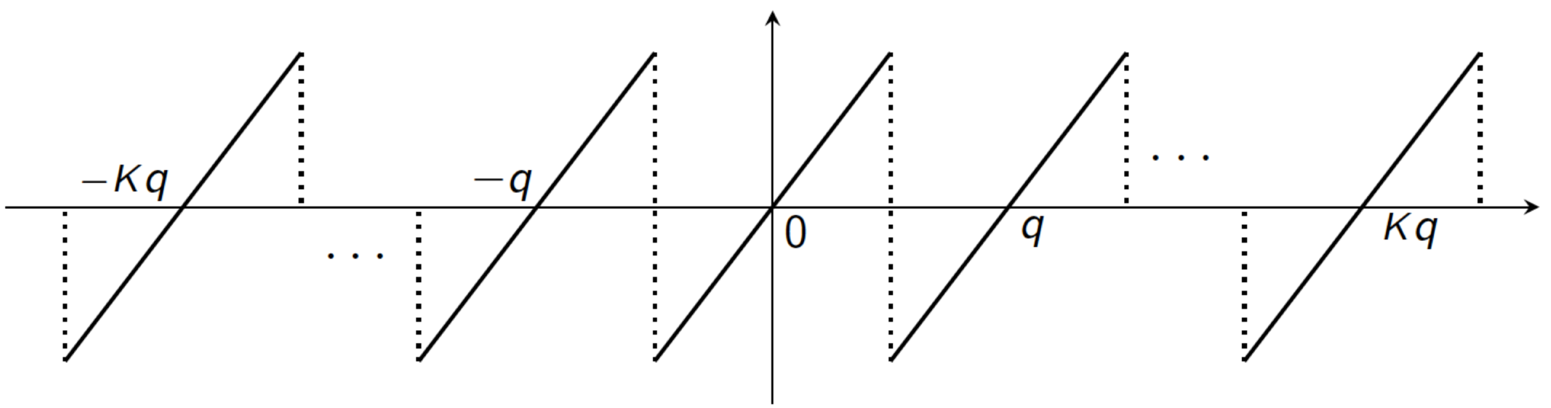

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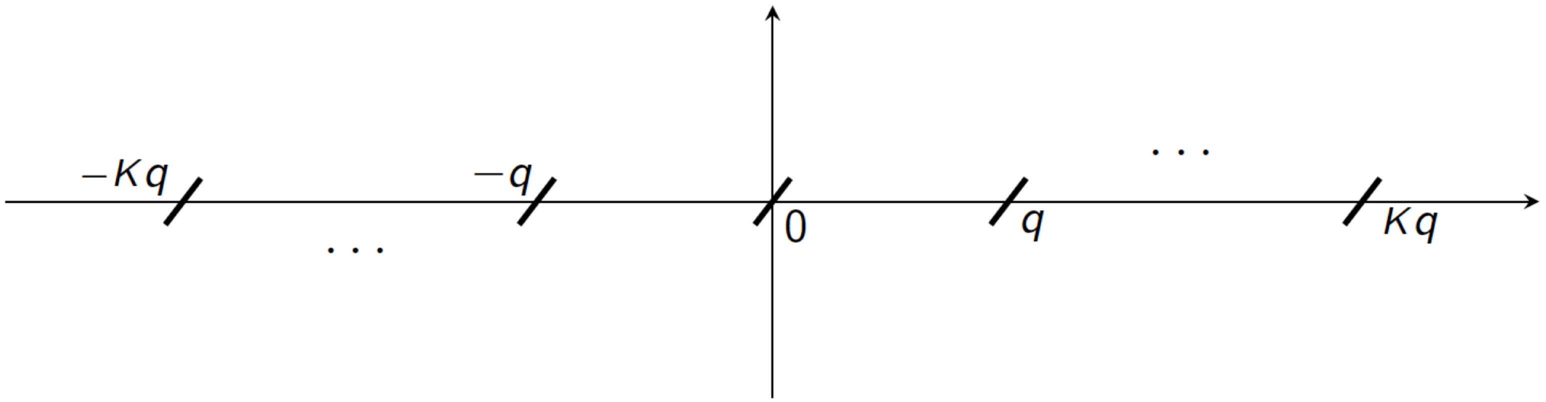

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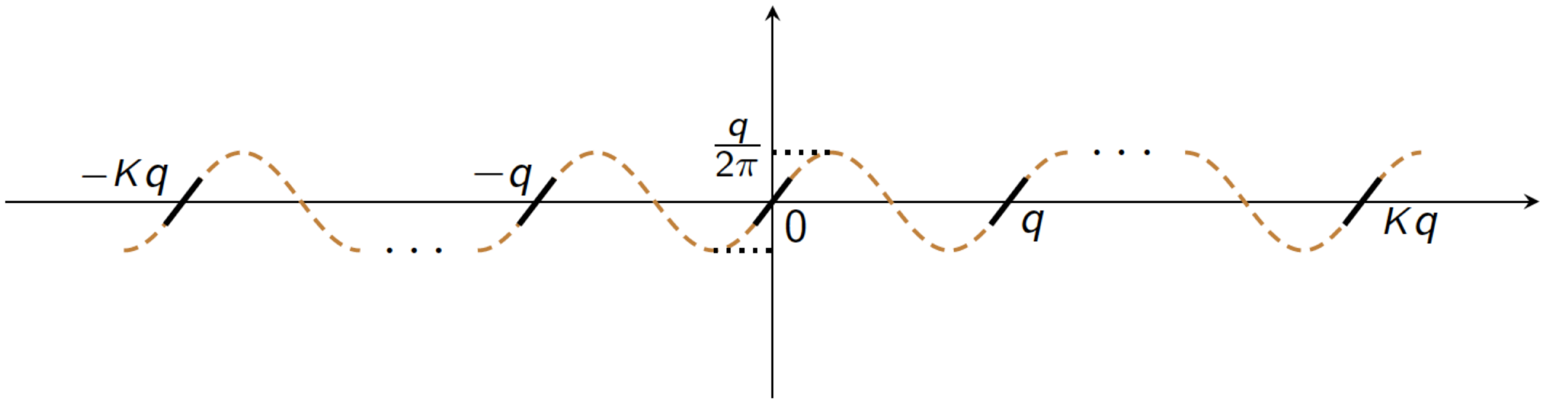
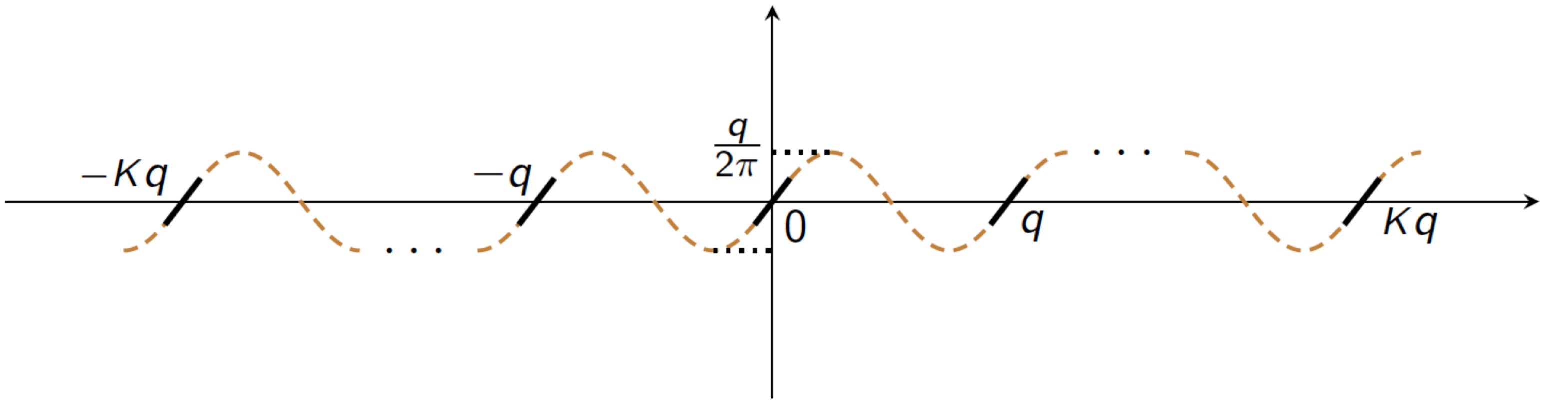
Use the formula $M \approx (q/2\pi) \cdot \sin\left[(2\pi/q)\,(q \cdot t + M)\right]$.



Evaluation of Sine

- Goal: Evaluate $M \approx (q/2\pi) \cdot \sin\theta$ for $\theta = (2\pi/q)\,(q \cdot t + M)$ such that $|\theta| < 2\pi K$



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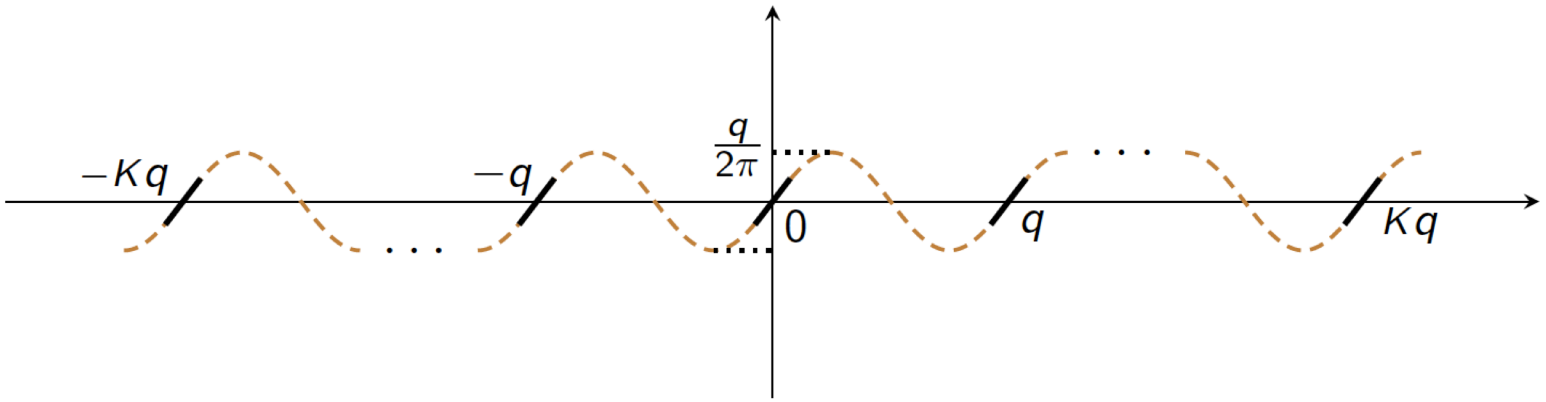

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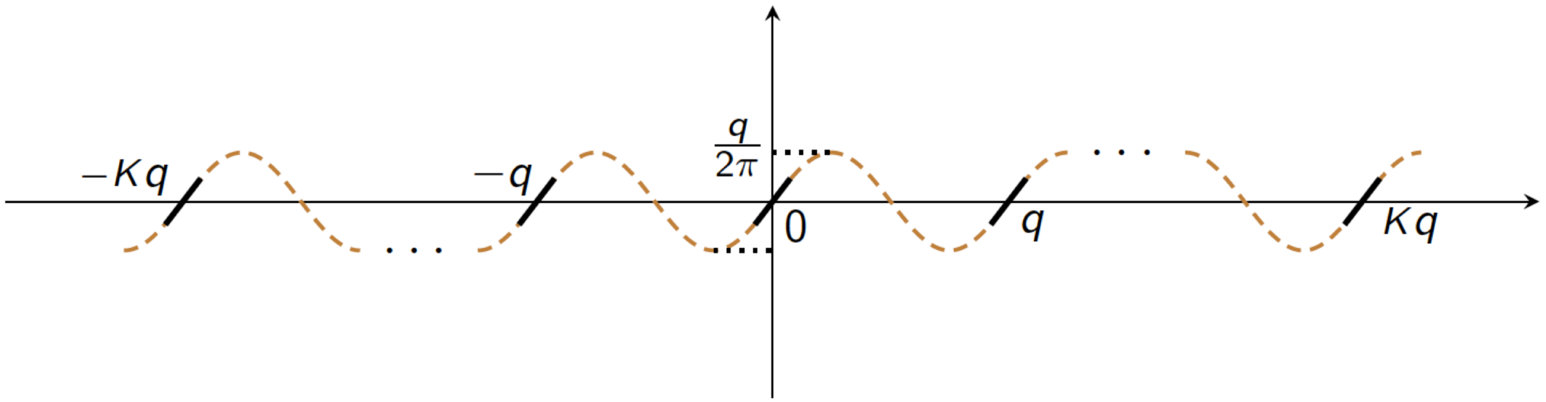
Degree	$d = O(Kq)$ to achieve $R_d = O(1)$.
Complexity	$O(Kq)$ operations.



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  Idea 3: Double-angle formula

  $\cos\theta = \cos^2(\theta/2) - \sin^2(\theta/2)$, $\sin\theta = 2\cos(\theta/2) \cdot \sin(\theta/2)$.

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- Efficiency

  Depth: $L = r + O(1) = O(\log(Kq))$.

  Complexity: $O(L)$ operations. Linear on the depth!

Summary

- $ct = Enc(M) \pmod q$ is an encryption of $(q \cdot t + M)$ in a large modulus.
- Approximation of Modular reduction $(q \cdot t + M)_q = M$ using a trigonometric function.
- Recursive evaluation strategy to reduce the computational costs.

Summary

- $ct = Enc(M) \pmod q$ is an encryption of $(q \cdot t + M)$ in a large modulus.
- Approximation of Modular reduction $(q \cdot t + M)_q = M$ using a trigonometric function.
- Recursive evaluation strategy to reduce the computational costs.

  ✓ No Bootstrapping Key.

  ✓ Linear Complexity on the depth $L = O(\log(|sk|_1 \cdot q))$ of decryption circuit.

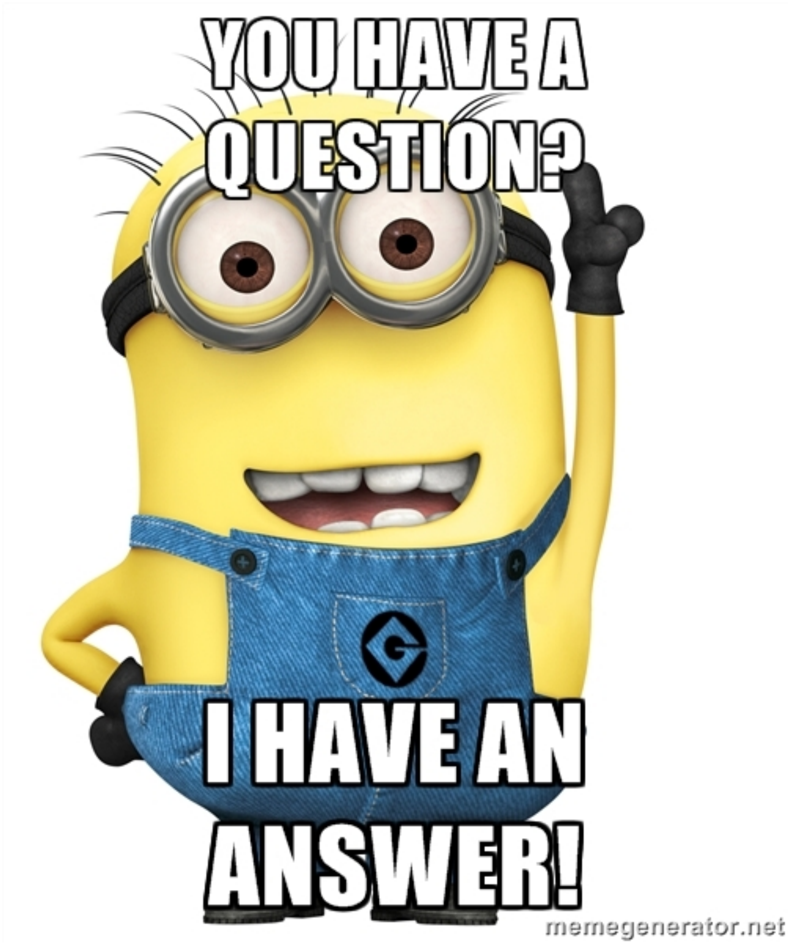
  ✓ Small Memory : 1 ciphertext encrypting $\exp(i \cdot \theta) = \cos\theta + i \sin\theta$.

  ✓ Implication : Machine Learning, Cyber-Physical System

Comparison & Experimental Results

HS15, CH18	Coeff To Slots $\tilde{O}(1)$ per slot	Bit/Digit Extraction	Slots To Coeff $\tilde{O}(1)$ per slot
Ours		**Sine Evaluation**	
DM15,CGGI16	Accumulator: O(n) operation / 1 slot		

- Digit Extraction: 6s (Z_{127}). 30s (Z_{127^2}). 15s (Z_{2^6}). 239s (Z_{2^8}).
- **Sine Evaluation**: 12.5s (12-bit precision). 68s (24-bit precision).
  [Song-Han-Kim-Kim-Cheon 18] Full Residue Number System: 8x ~ 12x speedup
- Accumulator: 0.06s (1 bit). 10s (6 bits)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